Remarks on the Calculation of the Enhancement Factor in the $K \to 2\pi$ Decay

Tran N. Truong

Centre de Physique Théorique, *

Ecole Polytechnique

F91128 Palaiseau, France

August 2000

Abstract

The problem of calculation of the enhancement factor due to the strong pion final state interaction is reexamined in the light of recent interest in understanding of the origine of the $\Delta I = 1/2$ rule and calculations of the CP violation parameter ϵ'/ϵ. It is shown that the traditional method of calculating the *absolute* enhancement factor is model dependent, while the method of relating $K \to 2\pi$ amplitude to the $K - \pi$ matrix element using Current Algebra is on a safer ground.

*unité propre 014 du CNRS

The problem of calculating the absolute enhancement factor due to the final state interaction is a well-known problem in particle physics [1, 2]. The solution for the hadronic final state interaction problem, if it existed at all, could be quite useful in understanding many problems in the low energy particle physics, in particular the origine of the $\Delta I = 1/2$ rule in the Kaon decay and the ratio ϵ'/ϵ in the CP violation of the $K \to 2\pi$ decays [3]-[8].

There are two approaches in the litterature. One is the classical potential method, using the well-known result of the correction due to the Coulombian final state interaction in β decays [1, 2]. In this approach, the interacting wave function is calculated at the origin due to the relatively long range strong interaction of the two hadrons in the final state compared with the shorter range production or decay mechanisms. This method is called, in the following, the potential approach. Here the *absolute* enhancement factor is calculated.

Another approach [3]-[8] to this problem is not to calculate the absolute enhancement factor, but only to calculate the energy variation of the enhancement factor between two energy scales which are not far apart. Using current algebra low energy theorem [9] or the effective Lagrangian, one process can be related to another.

For example, the $K \to 2\pi$, $K \to 3\pi$ are related to the $K - \pi$ mixing. Similarly the $\eta \to 3\pi$ rate is related to the $\eta - \pi$ mixing; the $K \to \pi\pi e\nu$ (K_{l4}) is related to $K \to \pi e\nu$ (K_{l3}) and $K \to e\nu$ (K_{l2}) amplitudes. In our problem, the $K \to 2\pi$ is related to the $K - \pi$ mixing. The latter process, having no final state interaction, can be calculated by the numerical method of the lattice gauge theory or by some approximation schemes. In the following, for simplicity, this method is called as the current algebra approach.

The purpose of this note is to compare these two approaches for $K \to 2\pi$ decay. It is pointed out that these apparently unrelated approaches are in fact similar, but the potential approach is unreliable while the current algebra is more believable. They both describe the energy variation of the Omnes function [10] or enhancement factor at two energy points. In the potential approach, it is the variation of the (unsubtracted) Omnes function at infinite energy, compared with the Omnes function evaluated at the Kaon mass is calculated. In the current algebra approach, the variation of the enhancement factor between two points, the soft current algebra and the physical matrix element points, is calculated; they are relatively near which makes this type of calculation reliable.

1 Absolute Enhancement Factor in $K \to 2\pi$ Decay

Let us first examine the potential approach. The effect of the final state interaction on the matrix element is just the wave function of the interacting final state hadrons evaluated at the origin.

We first want to show that the wave function at the origin does indeed have the strong interacting phase, it has the *unsubtracted* Omnes form, and then we want to show that its magnitude is, however, model dependent.

Under some restrictions on the conditions on the integral of the interacting potential $V(r)$ as a function of the distance, namely:

$$\int_0^\infty dr r^m V(r) < \infty \tag{1}$$

where $m = 1$ and $m = 2$. These are the conditions on the short range behavior of the potential. Under these restrictions, some rigourous results are known on the wave function at the origin.

The wave function at the origin or the enhancement factor is given by the inverse of the Jost function $f(-k)$, where k is the the relative momentum of the two-body system [11]. In the energy plane, $s = 4(m_\pi^2 + k^2)$, the inverse of the Jost function is analytic in the cut plane with a branch point at $s = 4m_\pi^2$; the zeros of the Jost function are on the real s axis and correspond to the bound states. The phase of the inverse of the Jost function, $1/f(-k)$, is the two-body elastic phase shifts δ. For $s \to \infty$, the inverse of the Jost function has a well-defined limit: it is real and equal to unity.(For a review of the potential theory see [12, 13, 14]). Because of these conditions, the inverse of the Jost function, which is denoted in the following as $1/D(s)$, has the following integral representation (assuming that there are no bound states):

$$\frac{1}{D(s)} = \exp(\frac{1}{\pi}\int_{4m_\pi^2}^\infty \frac{\delta(z)dz}{z - s - i\epsilon}) \tag{2}$$

This equation can be derived by applying the Cauchy theorem for $\ln(1/D(s))$, using the fact that there are no bound states and hence the Jost function $f(-k)$ has no zeroes in the s plane and is equal to unity at infinity. The convergence of the integral in Eq. (2) is assured by the restriction of the potential which demands that the phase shifts tend to zero sufficiently fast as $s \to \infty$. At an energy s, the enhancement factor is just given by Eq. (2). The physical value of the matrix element is obtained by setting $s = m_K^2$ where m_K is the Kaon mass. (If there were bound states, the expression for the wave function would be slightly more complicated and the variation of the phase shifts between zero and infinite energy must obey the Levinson's theorem [15], $\delta(0) - \delta(\infty)$=(number of bound states)π, with $\delta(\infty) = 0$). Because there are no bound states in $\pi\pi$ scattering, the S-wave $I = 0$ phase shift at the 2π threshold is zero and has to go to zero at $s = \infty$ sufficiently fast.

From these considerations, a calculation of the absolute enhancement factor implies implicitly a variation of the $1/D(s)$ factor between $s = \infty$ where it is unity, and $s = m_K^2$. This is a very large energy region to calculate the variation of the enhancement factor, and hence it is difficult to get a reliable result.

This sensitivity can also be seen from the study of the scattering of two particles by a potential. Although this problem was studied by the authors of the reference [1] who claimed that the enhancement factor is insensitive to the constructed potentials, we wish to point out that the two potentials that they constructed are quite similar, one is the

square well and the other is Gaussian. Because they are quite similar in both the short and long range behaviors, one cannot claim the insensitivity of the calculation of the enhancement factor.

In fact, it is well-known that the value of the wave function at the origin depends strongly on the behavior of the potential near the origin but the low energy behavior of the phase shifts does not. One can easily construct a potential with an inner repulsive core but with an outer attractive part to give approximately the same scattering length and effective range as those given by a purely attractive potential. Yet the wave functions at the origin are quite different for both cases. In the purely attractive potential the modulus of the enhancement factor is larger than unity. In the case of a more complicated potential, with an inner repulsive core and an outer attractive part, the modulus of the enhancement factor could be less than unity. In fact with an infinite repulsive core, the wave function at the origin is zero.

This result can also be seen from the integral representation of the inverse of the Jost function in terms of the elastic phase shifts, i.e from the *unsubtracted* form of the Omnes function, Eq. (2). It is clear that the enhancement factor evaluated at $s = m_K^2$ is sensitive to the asymptotic behavior of the phase shifts corresponding to the shorter range of the potential. A subtracted form of the Omnes function is less sensitive to the high energy behavior of the phase shifts due to the weight factor in the integral representation (see below).

Although some qualitative features of the potential approach may be correct, it is difficult to make the calculation scheme reliable because one has effectively to calculate the variation of the enhancement factor at $s = m_K^2$ compared to that at infinite energy (which is unity). Our viewpoint is therefore the unsubtracted form of the Omnes function or the absolute enhancement factor, being sensitive to the high energy behaviour of the phase shifts, should be avoided in theoretical calculations. It can however be used with confidence to study the energy dependence of the matrix element as will be shown below.

2 Current Algebra Low Energy Theorem: Relation between $K \to 2\pi$ and the off-shell $K - \pi$ transition

The above approach is in contrast with the calculation of the variation of the function $1/D(s)$ at low energy. This is a typical problem one has to deal with when Chiral Symmetry is relevant. To make this point clear, let us consider the chain K_{l4}, K_{l3}, K_{l2}. They are related to each other by the soft current algebra theorems and also by the Effective Lagrangian. Roughly speaking, in the limit of one the pion soft in the K_{l4} decay, its matrix element is equal to the K_{l3}. This relation is independent of whether there is a strong final state interaction between the two outgoing pions or not.

Using this idea, the problem of relating the $K \to 2\pi$ and the off-shell $K - \pi$ transition, was examined a long time ago [3, 4]. There are recently questions how various formula

in ref. [3, 4] were obtained [5, 7]. A more detailed explanation can be found in [8]. We briefly summarized here how these results were obtained. The effective Lagrangian for the $K \to 2\pi$ is given by:

$$M(K_S(k) \to \pi^+(p) + \pi^-(q)) = \frac{i}{\sqrt{2}} C f_\pi (2k^2 - p^2 - q^2) \tag{3}$$

and

$$M(K_L \to \pi^0) = -C\sqrt{2} f_\pi^2 q(\pi).q(K) \tag{4}$$

Hence Eqs. (3, 4) are related to each other by a Clebsch-Gordan coefficients and a factor of f_π in the limit of $p_\mu \to 0$. This is typically a current algebra low energy theorem. Unlike other current algebra results, the matrix elements given here are strongly energy dependent.

Let us consider the matrix element of $K(k) \to \pi^+(p) + \pi^-(q)$ as function of the complex variables $k^2 = s$ with first the two pion on their mass shell. This matrix element is an analytic function in s with a cut starting from $4m_\pi^2$ to ∞. The imaginary part of this matrix element can have contribution from the self energy graphs and also from the contribution from the unitarity relation $2\pi, 4\pi$,... intermediate states. Because it is an analytic function with a cut on the real axis from $4m_\pi^2$ to ∞, this matrix element can be expanded as a power series in $s - s_0$ where s_0 is outside the cut. Eq. (3) should be interpreted in this way and therefore includes the final state interaction effect.

Furthermore the Cabibbo-Gell-Mann Theorem [16] requires that the $K \to 2\pi$ matrix element has to vanish in the $SU(3)$ limit which is clearly satisfied by Eq. (3). Using this condition, one has to take the expansion point at $s_0 = m_\pi^2$ [3, 4, 8].

The $K \to 2\pi$ matrix element is therefore (with $k^2 = s$ and the pions are on their mass shell):

$$M(K_S(s) \to \pi^+\pi^-) = \sqrt{2} C f_\pi (s - m_\pi^2) \frac{1}{D(s, m_\pi^2)} \tag{5}$$

while the $K - \pi$ matrix element, with the pion on its mass shell, the Kaon off its mass shell and the weak hamiltonian carries no momentum, is:

$$M(K_L \to \pi^0) = -C\sqrt{2} f_\pi^2 m_\pi^2 \tag{6}$$

where

$$\frac{1}{D(s, m_\pi^2)} = \exp\left(\frac{s - m_\pi^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\delta(z) dz}{(z - m_\pi^2)(z - s - i\epsilon)}\right) \tag{7}$$

Eqs. (5, 6) give a relation between the $K - \pi$ and the $K \to 2\pi$ transitions. Physical value of $K \to 2\pi$ is obtained by setting $s = m_K^2$.

Because of the subtracted form of the Omnes function in Eq. (5), defined in Eq. (7), the enhancement factor is not sensitive to the high energy behavior of the phase shift (or to the inner part of the S-wave $\pi\pi$ potential). Here one studies the variation of the enhancement factor between two near-by points, $s = m_\pi^2$ and $s = m_K^2$.

3 Analogy with the Pion Form Factor Calculation

In the current algebra method, two apparently unrelated processes are related to each other by the current algebra soft pion theorem. Here the partially conservation of the axial current and the current commutation relations play an important role. Effective Lagrangian synthesizes these results in a simple manner.

It might be useful to compare the current algebra $K \to 2\pi$ problem with the corresponding vector pion form factor V(s) calculation.

The conservation of the hadronic vector current, together with the usual commutation relations in field theory, enables us to demonstrate the Ward identity at zero momentum transfer for the vector pion form factor $V(0) = 1$ [17]. This condition sets the scale for the form factor calculation. The calculation of the vector pion form factor is therefore similar to the relation between $K \to 2\pi$ and $K - \pi$ mixing. It is of interest to see how the $\pi\pi$ final state interaction effect due to the potential approach would give.

Because the two pions are in the relative P-state, the first derivative of the interacting wave function at the origin is relevant in calculating the final state interaction effect. The enhancement factor is given by the inverse of the P-wave Jost function. The low energy theorem for V(0) as required by the Ward identity is violated because the inverse of the Jost function at $s = 0$ is no longer equal to unity. It is however equal to unity at $s = \infty$ which is not required by any other general principle.

The potential approach to this problem has to be modified: one simply has to forget the condition at infinite energy, but taking into account of the Ward identity condition at zero momentum transfer:

$$V(s) = D_1(0)/D_1(s) \tag{8}$$

where $D_1(s)$ is similar to Eq. (2) with the S-wave phase shifts replaced by the P-wave phase shifts. Eq. (8) is the standard formula for the vector pion form factor $V(s)$.

4 Conclusion

In conclusion, the absolute enhancement factor for $K \to 2\pi$ is model dependent, but the relation between $K \to 2\pi$ and $K - \pi$ using analyticity, unitarity and the effective Lagrangian or current algebra, is more reliable.

References

[1] N. Isgur, K. Maltman, J. Weinstein and T. Barnes, Phys. Rev. Lett. **64** (1990) 161.

[2] G. E. Brown, J. W. Durso, M. B. Johnson and J. Speth, Phys. Lett. **238** (1990) 20.

[3] T. N. Truong, Phys. Lett. **B207** (1988) 495.

[4] T. N. Truong, Phys. Lett. **B313** (1993) 221.

[5] M. Neubert and B. Stech, Phys. Rev. **D44** (1991) 775.

[6] E. Pallante and A. Pich, Phys. Rev. Lett.**84** (2000) 2568.

[7] A. J. Buras, M. Ciuchini, E. Franco, G. Isidori, G. Martinelli and L. Silvestrini, Phys. Lett. **B480** (2000) 80.

[8] T. N. Truong (ph-hep/0004185).

[9] For a review, see *Current Algebra* by S. L. Adler and R. Dashen, W.A. Benjamin, New York 1968.

[10] N. I. Muskhelishvili, *Singular Integral Equations* (Noordhoof, Groningen, 1953). R. Omnes, Nuovo Cimento **8** (1958) 316.

[11] R. Jost, Helv. Phys. Acta. **22** (1947) 256.

[12] M. L. Goldberger and K. M. Watson, *Collision Theory*, John Willey and Sons, Inc., New York 1964.

[13] J. Gillespie, *Final State Interactions*, Holden-Day Inc., San Francisco 1964.

[14] R.Blankenbecler, M. L. Goldberger, N. Khuri and S. B. Treiman, Ann. of Phys. **10** (1960) 62.

[15] N. Levinson, Danske Videnskab. Selskab, Mat.-fys. Medd. **25**, No.9 (1949).

[16] N. Cabibbo, Phys. Rev. Lett. **12** (1964) 62; M. Gell-Mann, Phys. Rev. Lett. **12** (1964) 155. See also more recent references cited in [7].

[17] S. Weinberg, *Lectures on Elementary Particles and Quantum Field Theory, 1970 Brandeis University Summer Institute in Theoretical Physics*, MIT Press, Cambridge 1970.